When p *is odd:*

Number of $(p-1)$-dimensional hyperplanes	External Centers of Similitude	Internal Centers of Similitude
1	$\frac{p^2+p}{2}$	0
$p+1$	$\frac{p^2-p}{2}$	p
$\frac{(p+1)p}{2!}$	$\frac{p^2-3p+4}{2}$	$(p-1)2$
$\frac{(p+1)p(p-1)}{3!}$	$\frac{p^2-5p+12}{2}$	$(p-2)3$
$\vdots$	$\vdots$	$\vdots$
$\frac{(p+1)p(p-1)\cdots(p-k+2)}{k!}$	$\frac{p^2-(2k-1)p+(k-1)2k}{2}$	$(p-(k-1))k$
$\vdots$	$\vdots$	$\vdots$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+5}{2}\right)}{\left(\frac{p-1}{2}\right)!}$	$\frac{p^2-4p+3}{4}$	$\left(\frac{p+3}{2}\right)\left(\frac{p-1}{2}\right)$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+3}{2}\right)}{2\cdot\left(\frac{p+1}{2}\right)!}$	$\frac{p^2-1}{4}$	$\left(\frac{p+1}{2}\right)^2$

When p *is even*, the last two lines become:

$\frac{(p+1)p(p-1)\cdots\left(\frac{p+6}{2}\right)}{\left(\frac{p-2}{2}\right)!}$	$\frac{p^2+8}{4}$	$\left(\frac{p+4}{2}\right)\left(\frac{p-2}{2}\right)$
$\frac{(p+1)p(p-1)\cdots\left(\frac{p+4}{2}\right)}{\left(\frac{p}{2}\right)!}$	$\frac{p^2}{4}$	$\frac{p}{2}\left(\frac{p+2}{2}\right)$

HISTORICAL NOTE ON CENTERS OF SIMILITUDE OF CIRCLES.

By RAYMOND CLARE ARCHIBALD, Brown University.

In the paper on "Centers of Similitude of Circles and certain Theorems attributed to Monge. Were they known to the Greeks?"[1], I endeavored to show, especially through consideration of a problem in the book *On Tangencies*

[1] AMERICAN MATHEMATICAL MONTHLY, January, 1915, vol. 22, pp. 6–12. *Addenda:* In note 2, page 10, line 5, *for* cones *read* conics; on page 12, line 2, *for* side *read* sides. The two following historical notes may also be given:

(*a*) The theorem that "*The six centers of similitude of three coplanar circles lie by threes on four straight lines*" has been attributed to Monge who published it in 1798. Proof by analytical geometry that the three *external* centers of similitude are collinear was given by L. PUISSANT in his *Recueil de diverses propositions de géométrie*, Paris, 1801, pp. 50–56; it was probably here that the particular case of two of the circles being equal was first considered in recent times. No reference is made to Monge. There was a German edition by Hahn (Berlin, 1806). In the second French edition (Paris, 1809) mention is made (pp. 131–137) of Monge and the *general* theorem.

by Apollonius of Perga (about 225 B. C.), that several of the now well-known properties of centers of similitude of circles were also familiar to the Greeks. When writing the paper it did not occur to me to reinforce my argument by reference to another work by Apollonius, namely that *On Plane Loci*.[1] Pappus's account[2] of this work has been the basis of restorations and discussions by Fermat,[3] Schooten,[4] Simson,[5] Camerer,[6] Lhuilier,[7] Bonnycastle,[8] Breton (de Champ)[9] and Zeuthen.[10]

The *Plane Loci* consisted of two books which contained 147 propositions and figures, and 8 lemmas. Pappus states a general proposition, the detailed discussion of which he seems to indicate as the chief original contribution of Apollonius to the first book of his work.[11] This general proposition is as follows:

(*b*) To Monge (1798) is due the theorem (emended form of that given on p. 7 of the above mentioned paper): "*Given any four spheres in space, fixed in magnitude and position, and the six cones tangent to them in pairs, externally; then the six vertices lie in a plane and indeed on four straight lines in the plane." If the six other tangent cones be drawn then the twelve conical vertices lie by sixes in five planes.* (This theorem is referred to by Mr. Brown in the preceding paper.) Monge overlooked the fact that there were three other planes with a similar property. This seems to have been first remarked by J. B. Durrande in *Annales de mathématiques* (Gergonne), Juillet, 1820, tome 11, pp. 18–20. He there calls the 8 planes "plans de similitude" and also gives synthetic proof of Monge's theorems for any positions of circles and spheres. The proofs of Monge are only applicable for circles and spheres external to one another. Durrande also used the term "axes de similitude" in connection with collinear centers of similitude of circles and spheres (pp. 10 and 17).

[1] The most recent historical sketch is by G. Loria in *Le Scienze esatte nell' antica Grecia.* Seconda edizione totalmente reviduta. Milano, 1914. Pp. 393–396 and 440–443. A reference may also be given to G. S. Klügel's *Mathematisches Wörterbuch*, erste Abtheilung, dritter Theil, Leipzig, 1808. Pp. 697–698.

[2] Pappi Alexandrini, *Collectionis*, edidit F. Hultsch, vol. 2, Berolini, 1877, pp. 660–671, 852–865. In C. I. Gerhardt's Greek-German edition, Halle, 1871, pp. 20–29; 166–175.

[3] *Varia opera mathematica*, D. Petri de Fermat. Tolosae, 1679, pp. 12–43; facsimile edition, Berolini, 1861—E. Brassine's *Précis des Oeuvres mathématiques de P. Fermat*, Toulouse, 1853, pp. 39–41—*Oeuvres de Fermat*, Paris, tome 1, 1891, pp. 3–51; tome 3, 1896, pp. 3–48. See also tome 2, pp. 5, 30, 56, 74, and 100.

[4] *Exercitationum mathematicarum. Liber III. Continens Apollonii Pergaei Loco Plana restituta.* Lugd. Batav. 1656, pp. 191–292—*Derde Bouck der mathematische Oeffeningen begrijpende, Apollonii Pergaei herstelde Vlacke Plaetsen* . . . door Franciscus Van Schooten. Amsterdam, 1660, pp. 185–272.

[5] *Apollonii Pergaei Locorum Planorum Libri II. Restituti* a Roberto Simson. Glasguæ, 1749. 252 pp.

[6] *Apollonius von Pergen ebene Oerter.* Wiederhergestellt von Robert Simson . . . Aus dem Lateinischen übersezt, mit . . . Aufgaben begleitet von J. W. Camerer. Leipzig, 1796. 455 pp. + 17 plates.

[7] *Éléments d'analyse géométrique et d'analyse algébrique appliquées a la recherche des lieux géométriques.* Par S. Lhuilier. Paris, 1809. "Lieux traités par Apollonius, suivant Simson" and "additions diverses aux lieux plans d'Apollonius," pp. 35–111.

[8] J. Bonnycastle, *Elements of Geometry.* Sixth edition, London, 1818. "De Locis Planis," pp. 371–375.

[9] *Recherches nouvelles sur les porismes d'Euclide.* Paris, 1855. Pp. 91–95. Also in *Journal de mathématiques pures et appliquées*, tome 20, 1855. Pp. 299–303.

[10] H. G. Zeuthen, *Die Lehre von den Kegelschnitten im Altertum.* Deutsche Ausgabe von R. v. Fischer-Benzon. Kopenhagen, 1886. Pp. 207–212.

[11] The early propositions of the *second* book include the following familiar results: (1) The locus of points, the difference of the squares of whose distances from two fixed points is constant, is a straight line perpendicular to the straight line joining the points; (2) The locus of the points,

"Two straight lines are drawn either from the same fixed point or from two fixed points, in the same direction or in such a way as to form a fixed angle; the lengths of these lines are in a constant ratio to one another or their rectangle is constant. If the extremity of one of them describes a plane locus given in position, the extremity of the second will also describe a plane locus, given in position, which is either of the same or of different species from the first."

A particular proposition included in this general one may be formulated as follows:

Through a point O draw lines $OP_1, OP_2, OP_3, \cdots$ to the various points $P_1, P_2, P_3, \cdots$ on a circle.[1] *Divide $OP_1, OP_2, OP_3, \cdots$ internally at $Q_1, Q_2, Q_3, \cdots$ respectively, and such that $OP_1 : OQ_1 = OP_2 : OQ_2 = \cdots =$ a const., and externally at $R_1, R_2, R_3, \cdots$ respectively, such that $OP_1 : OR_1 = OP_2 : OR_2 = \cdots =$ a const. Then the locus of the Q's is a circle,*[1] *and the locus of the R's is a circle.*[1]

Here O is the external center of similitude of the circles (P) and (Q), and the internal center of similitude of the circles (P) and (R). It is exactly such a proposition which Simson and others (*l. c.*) consider in their restorations in various cases when the circles (1) are exterior to one another; (2) intersect; (3) are such that one is inside of the other. The property of parallel radii joining corresponding points of the pairs of circles, arises in the course of the proofs.

My earlier argument that Apollonius was familiar with the centers of similitude of circles and some of their chief properties has thus been reinforced through consideration of another of his works.

A CIRCLE THEOREM.

By ROGER A. JOHNSON, Adelbert College, Western Reserve University.

THEOREM. *If three equal circles are drawn through a point, the circle through their other three intersections is equal to each of them.*

Proof. Denote the centers of the circles (see figure 1 of the next paper) by C_1, C_2, C_3, the intersections of C_2 and C_3 by O and P_1, those of C_3 and C_1 by O and P_2, those of C_1 and C_2 by O and P_3. Then $OC_2P_1C_3$ is a rhombus, and so is $OC_3P_2C_1$. Hence, C_2P_1 and C_1P_2 are equal and parallel, $C_1C_2P_1P_2$ is a parallelogram, and P_1P_2 is equal to C_1C_2. Thus the triangles $C_1C_2C_3$ and $P_1P_2P_3$ are congruent, and have equal circumcircles. But the circumcircle of the former has its center at O, and is equal to each of the given circles. Hence, the circle through P_1, P_2, P_3 is equal to each of the given circles.

the ratio of whose distances from two fixed points is constant, is either a straight line or a circle —the Circle of Apollonius. Eutocius gives the proof of Apollonius for this latter locus (Apollonius, ed. Heiberg, Vol. 2, pp. 180–185). The name "Circle of Apollonius" is, however, a misnomer, since the construction of this locus connected with his name appears in exactly the same form at a much earlier date in Aristotle's *Meteorologica*, III, 5, 376 f.

[1] "Circle" is here considered as a curved line. The cases of this proposition when we substitute "straight line" for "circle" were discussed by Euclid in Propositions 35–36 of his *Data* (Simson's edition, Prop. 39—for example: *Elements of Elucid . . . also Euclid's Data*, 9th ed., Edinburgh, 1793, pp. 393–394).